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Development of Arithmetic Fluency: A Direct Effect of Reading Fluency?

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The present longitudinal study investigated the developmental trajectories of addition, subtraction, and multiplication fluency; their relationships to general cognitive functions; and potential interrelations between reading fluency and growth in arithmetic fluency for different operations. The central prediction was that measures of arithmetic fluency for different operations would be differentially related to reading fluency because of variations in the timing and format of arithmetic instruction. Panel data on children's arithmetic fluency and reading fluency performance were collected twice per year in third, fourth, and fifth grades ($N = 449$). For each operation tested (addition, subtraction, and multiplication), a series of bivariate (reading/arithmetic) latent growth and autoregressive latent trajectory models were constructed. After controlling for background cognitive variables, significant short- and long-term relationships to reading fluency were observed only for multiplication fluency. Specifically, reading fluency in early third grade predicted both immediately subsequent multiplication fluency scores and the slope of growth in multiplication fluency across subsequent grade levels. Exploratory analyses of covariate effects showed that whole number line estimation accuracy was related to growth in multiplication and subtraction fluency, while general verbal ability had a unique relationship to subtraction fluency that was not observed for either addition or multiplication. Overall, results indicate that children's level of reading fluency and differences among arithmetic operations should be considered both when conducting research and when making instructional decisions about how to build students' arithmetic fluency.

Educational Impact and Implications Statement

This study investigates the relationship between reading fluency and the development of arithmetic fluency—in particular, how this relationship may differ depending on arithmetic operation. A large, representative sample of U.S. public school students completed tests of reading fluency and addition, subtraction, and multiplication fluency, respectively, twice per year from third to fifth grade. Statistical models of growth in reading fluency and arithmetic fluency for each operation showed that early reading fluency predicted both short- and long-term growth in multiplication fluency but did not predict growth in addition or subtraction fluency. These results suggest that the abilities underlying arithmetic fluency are not the same for all operations; both researchers and practitioners should consider the increased importance of reading fluency for the process of learning multiplication facts, which differs in both timing and instructional format from the process of learning addition and subtraction facts earlier in school.

Keywords: arithmetic fluency, reading fluency, multiplication facts, instructional methods, autoregressive latent trajectory model

Arithmetic fluency is critical for later mathematics learning (Cowan et al., 2011; National Mathematics Advisory Panel, 2008). Speed and accuracy in simple computation predict the subsequent development of strategies for not only more complex arithmetic (Carr & Alexeev, 2011) but also fractions understanding (Hecht & Vagi, 2010; Jordan et al., 2013) and prealgebraic reasoning (Powell, Kearns, & Driver, 2016). Children with mathematics difficulties (MDs) exhibit poor fluency with basic addition and subtraction facts (Jordan, Hanich, & Kaplan, 2003), limiting the availability of cognitive resources for more complex problem solving (Locuniak & Jordan, 2008). Although most research has focused on fluency with single-digit addition and subtraction, children with MDs also exhibit poor multiplication fluency (Mabbott & Bisanz, 2008). Poor multiplication fluency has been linked to later difficulty with fractions (Hansen et al., 2015), an important predictor of high school mathematics achievement (Siegler et al., 2012).

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Potential sources for children's struggles with arithmetic fluency—particularly those of children with MDs—have been explored in previous research. Poor arithmetic fluency has been associated with deficits in the central executive domain (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007), including limitations to working memory and heightened interference effects (Bailey, Hoard, Nugent, & Geary, 2012; De Visscher & Noël, 2014). Arithmetic fluency is also related to poor number processing, including weak and indistinct representation of arithmetic problems in the intraparietal sulcus (Ashkenazi, Rosenberg-Lee, Tenison, & Menon, 2012).

Reading difficulties, such as poor phonological processing, may likewise undermine arithmetic fluency, although findings are mixed. Geary (1993) argued that arithmetic facts may be primarily encoded in terms of phonemic features, and Fuchs et al. (2005) found that phonological processing predicted fluency with addition facts in first grade. Korpipää et al. (2017) found that phonological processing predicted time-specific covariation between reading ability and addition/subtraction performance in first grade but not seventh grade. However, Jordan et al. (2003) failed to find a meaningful relationship between mastery of arithmetic facts and reading abilities, noting that children with reading difficulties (RDs) but not MDs exhibit typical fact fluency, whereas children with MDs exhibit poor fact fluency even in the absence of RDs (Hanich, Jordan, Kaplan, & Dick, 2001). This suggests that poor arithmetic fluency derives primarily from difficulty accessing and manipulating numerical representations. However, as Jordan et al. themselves noted, research on arithmetic fluency has focused primarily on addition and subtraction, but multiplication fluency may be more strongly associated with reading abilities (Cohen, Dehaene, Chochon, Leherico, & Naccache, 2000).

A few previous studies have hinted at a unique relationship between reading fluency and multiplication fluency, but each has limitations that make it difficult to draw firm conclusions. Tieche Christinat, Conne, and Gaillard (1995) found that children with specific language impairment exhibited more errors in single-digit calculations than did matched controls, but only for multiplication. However, this research might not generalize to typical populations. Meanwhile, work in neuropsychology suggests that fluency in multiplication may be especially dependent on reading ability. Prado, Mutreja, and Booth (2014) found that activity in a left temporal cortex region associated with language increased with grade level for multiplication but not subtraction. Generalization is an issue here as well, though, because this study did not include children with poor reading or mathematics skills. To reach more definitive conclusions about how differences in operations may alter relations between reading and arithmetic fluency in broad populations, researchers need behavioral evidence from longitudinal studies with large, representative samples of schoolchildren. A further question that remains open is whether reading fluency influences the acquisition of multiplication facts in a direct, unmediated manner, especially when facts are memorized, in large part, from written lists or tables. Reading fluency is a measure of the speed and accuracy with which text can be read; if some children read more slowly or inaccurately, this may undermine learning of arithmetic facts from written materials.

Although recent behavioral research has investigated how reading fluency and arithmetic fluency are related, the focus has not been direct effects of the former on the latter, nor has such work distinguished among arithmetic operations. For instance, Koponen, Salmi, Eklund, and Aro (2013) and Koponen et al. (2016) analyzed the relationship between reading fluency and arithmetic fluency, but their main focus was the possibility of a "common cause" relationship wherein underlying cognitive factors predicted both forms of fluency. These studies found that after controlling for other cognitive variables, rapid automatized naming (RAN) and counting ability each accounted for significant variance in both reading fluency and arithmetic (addition and subtraction) fluency. This led Koponen et al. to conclude that individual differences in RAN and counting ability produce an observed relationship between reading fluency and arithmetic fluency. Similarly, Korpipää et al. (2017) found that RAN, counting ability, letter knowledge, working memory, and nonverbal reasoning accounted for most of the covariation between arithmetic fluency and reading fluency in both first grade and seventh grade, suggesting that the relation between the two forms of fluency is mostly time-invariant. However, multiplication fluency was only tested in seventh grade (along with division), and the fluency measure did not distinguish among operations. Thus, although obviously important, such studies do not address whether reading fluency has a direct effect on arithmetic fluency or whether the relation between reading fluency and arithmetic fluency differs across operations.

Broadly speaking, the goal of the present study was to examine how reading fluency relates to addition, subtraction, and multiplication fluency, respectively. As outlined in the next section, there are a number of reasons to think that the relationship between reading fluency and arithmetic fluency may differ for multiplication relative to addition and subtraction. Although we had no a priori expectations regarding differential relationships to reading fluency for addition versus subtraction fluency, any effects that emerge could likewise have important implications for our understanding of the processes that underlie the development of arithmetic fluency. Given the documented relations between arithmetic fluency and MDs, our results may also point toward factors that contribute to comorbidity between MDs and RDs (e.g., Willcutt et al., 2013). Finally, the findings of this study may have important implications for the design of instruction to build students' fluency with different arithmetic operations.

Learning Different Arithmetic Operations

In the United States, the primary difference between how addition and subtraction facts are learned versus how multiplication facts are learned relates to when these facts are taught and how they are committed to memory. Although the Common Core State Standards (CCSS) do not recommend that arithmetic facts be learned via any particular method, simple addition and subtraction facts (combinations to 10) are typically taught starting in kindergarten and ending in first grade (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). Many children are exposed to simple addition and subtraction facts more informally before kindergarten (e.g., at home and/or day care; Jordan, Kaplan, Oláh, & Locuniak, 2006), but regardless, the same overarching constraint applies across this entire age range—children cannot yet read proficiently and, therefore, are not expected to begin representing sums and differences using written symbolic expressions until the end of kindergarten, after combinations to 10 have already been taught (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

Instead, children first learn basic number combinations using meaningful nonsymbolic representations (i.e., manipulatives, drawings) and simple strategies such as "counting on" from the larger addend (Baroody, 1985; Jordan, Kaplan, Ramineni, & Locuniak, 2009). Useful examples come from the CCSS-aligned curricula EngageNY and Eureka Math, which are nearly identical, share the same publisher (Great Minds, Washington, DC), and jointly represent the most commonly used elementary mathematics curriculum in the United States (Opfer, Kaufman, & Thompson, 2016). Composition/decomposition of sums to 10 is taught using concrete and pictorial representations, along with strategies such as "counting on." Addition through 20 is addressed starting in second grade, but children are taught to solve such problems largely by extrapolating from what they know about sums through 10 rather than through rote recall. For example, children learn to use the fact that $5 - 3 = 2$ to help them solve the problem $15 - 3 = 12$.

In contrast, EngageNY and Eureka Math do not introduce multiplication until third grade. Although the CCSS suggest that multiplication should be initially taught in terms of groups of objects (e.g., four groups of three objects), children are expected to "know from memory all products of two one-digit numbers" by the end of third grade. Accordingly, although EngageNY and Eureka Math show students how to "construct" products through repeated addition, skip counting, and the use of the distributive or commutative properties, subsequent instruction and practice are largely aimed at memorization of the products of single-digit numbers. Multiplication facts are regularly presented as written arithmetic expressions in patterned groups associated with particular factors (i.e., 5s, 7s, 9s). This leads the process of learning multiplication tables—even in a curriculum that intentionally departs significantly from the strict focus on rote memorization typical of past conventional instruction—to necessarily be much more dependent on children's reading ability compared with the process of learning addition and subtraction facts. Such repetitious practice is much less feasible for learning addition and subtraction facts in earlier grades, not only because of younger children's more limited reading abilities but also because children's ability to engage in rehearsal as an explicit memorization strategy typically does not emerge until approximately Age 7, when most children are in second grade (Henry & Millar, 1993).

Failure to Master Multiplication Facts

Difficulties mastering multiplication facts may be both more frequent and more consequential than difficulties with addition and subtraction facts. An inability to immediately retrieve answers is more common for multiplication, because the set of facts students need to memorize is much larger. Failure to improve in multiplication has been associated with a reversion from the use of retrieval to the use of numerical operations as a backup strategy (Suárez-Pellicioni, Prado, & Booth, 2018). However, backup strategies for multiplication (e.g., repeated addition) are less efficient than backup strategies for addition and subtraction (e.g., finger counting, counting on from an addend, decomposition). Thus, rote memory for facts is more important for multiplication fluency than for addition and subtraction fluency.

Prior research has shown that mastery of multiplication facts depends on the number of repetitions during the learning process, and complicating matters, the number of required repetitions to achieve mastery is greater for children with MDs (Burns, Ysseldyke, Nelson, & Kanive, 2015). Poor reading fluency may compound this difficulty, as the number of symbolic representations that can be read within a given length of time is decreased, and the time needed to complete a given number of repetitions is increased. Although there are relatively easy means for quickly reaching answers to unmemorized single-digit addition and subtraction problems, this is not true for multiplication, meaning poor reading fluency may have a greater effect on response times for calculations of products.

The Present Study

Our central question was whether measures of arithmetic fluency across operations are differentially related to reading fluency, because of variations in task demands and the timing and format of arithmetic instruction. We used panel data collected from children over six time points between third and fifth grade and constructed a series of bivariate longitudinal models linking reading fluency (based on a test of single-word reading speed) to fluency with multiplication, addition, and subtraction facts, respectively. We then identified the best-fitting model for each operation to observe relationships to reading fluency. The set of statistical models tested allowed not only for investigation of long-term effects of early reading fluency on the trajectory of arithmetic fluency growth but also for investigation of shorter term effects of reading fluency on arithmetic fluency across consecutive time points.

We expected that after controlling for background cognitive variables, any direct effect of reading fluency would be stronger and more persistent over time for multiplication fluency compared with addition and subtraction fluency. Although we cannot rule out the possibility that reading fluency contributes to early learning of addition and subtraction facts through formal and informal interactions prior to entry into third grade, any relationships between early reading fluency (at the start of third grade) and long-term growth in addition and subtraction fluency would be evident in our longitudinal models. Although there may be growth in addition and subtraction fluency, a lack of persistent relationships to early reading fluency would provide evidence that any observed effects of reading fluency on growth in multiplication fluency are driven by greater dependence on written materials during instruction. Although we had no specific predictions regarding potential effects of cognitive covariates related to mathematics achievement (i.e., nonverbal reasoning, verbal ability, working memory, attentive behavior, and whole number line estimation), we nonetheless explored whether such variables predict children's starting points and growth trajectories for reading and arithmetic fluency measures.

Our study makes three primary and important contributions to the research literature. First, to our knowledge, no prior work has longitudinally investigated whether relationships between reading fluency and arithmetic fluency derive from a direct effect of the former on the latter rather than a common cause associated with underlying cognitive variables. Second, although previous work may suggest a unique relationship between reading fluency and multiplication fluency, such work was not conducted with broad school populations and did not consider the possibility that differential relationships to reading fluency across operations mirror differences in the timing and form of instruction for multiplication versus addition and subtraction. Third, no prior work has comprehensively examined differences across arithmetic operations with respect to fluency growth trajectories, relationships to reading fluency, and covariate effects. A better understanding of *how* fluency for different arithmetic operations develops will enable researchers and educators to construct new ideas for pedagogy and curricula that improve children's arithmetic fluency.

Method

Participants

Participants were originally recruited from nine socioeconomically diverse elementary schools in two adjacent school districts for a large longitudinal study of fraction learning. All research was approved by the Institutional Review Board at the University of Delaware. The sample for the present analysis included 449 participants (234 female), with a mean age of 105.9 months at the first time of measurement in third grade. Low-income students (identified by participation in a free/reduced school lunch program) comprised 60% of the sample, and 11% were English language learners (ELLs). The sample included all children with complete covariate information who completed dependent measures at one or more time points. This group was 53% White, 38% Black, 6% Asian, and 3% American Indian/Alaska Native, with 16% further identifying as Hispanic. Data on background variables was collected during third grade, whereas measures of reading fluency and arithmetic fluency for addition, subtraction, and multiplication were administered twice per year in third, fourth, and fifth grade, respectively.

Starting with the adoption of the CCSS prior to the second year of our study (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), participating students received the Math Connects curriculum published by Macmillan/McGraw-Hill, Columbus, OH. Like the EngageNY and Eureka Math curriculum, Math Connects is well-aligned with the CCSS for arithmetic instruction. Multiplication is not introduced until third grade, and multiplication facts are taught through essentially the same methods described earlier for EngageNY and Eureka Math. Nearly all of the instruction that participants received prior to joining our study occurred before the implementation of the CCSS, with most students receiving the Investigations in Number, Data, and Space curriculum (TERC, 1998). This was among the most widely used "reform mathematics" curricula aligned with standards set (and later discarded) by National Council of Teachers of Mathematics (1989). One of the central features of reform mathematics curricula was the deemphasis of computation and memorization in favor of more "meaningful" forms of problem solving (Baxter, Woodward, & Olson, 2001). Thus, we can be confident that before joining our study, participants tended to receive addition and subtraction instruction that did not focus on extensive practice with written symbolic representations.

Measures

Arithmetic fluency. To measure arithmetic fluency in addition, subtraction, and multiplication, we administered separate subtests of the Wechsler Individual Achievement Test-III (WIAT; Wechsler, 2009) corresponding to each of these operations. Each paper-and-pencil subtest presents a set of problems with a given operation and prompts children to solve as many as they can within a 60-s time limit. For addition and subtraction, sums and minuends were 18 or less. Multiplication problems included factors up to 12. Our analyses used the raw score (total number correct) at each time point to best illustrate growth over time within individuals. Wechsler (1992) reports split-half reliabilities at fifth grade for the addition, subtraction, and multiplication subtests of .87, .91, and .90, respectively.

Reading fluency. The measure of reading fluency was the Sight Word Efficiency subtest from the Test of Word Reading Efficiency (TOWRE; Torgesen, Rashotte, & Wagner, 1999). This test presents a list of 104 written words and prompts children to read as many words out loud as possible within 45 s. As with arithmetic fluency, we used the raw score at each time point, which is the total number of words correctly read within the time limit. Test–retest reliability for this measure is .97 for children aged 6 to 9 years (Torgesen et al., 1999).

Background variables. We included in our model a variety of background variables (assessed in third grade) to control for general cognitive abilities and demographic factors.

Nonverbal reasoning. Nonverbal reasoning ability was assessed using age-scaled scores from the matrix reasoning subtest of the Wechsler Abbreviated Scale of Intelligence (Wechsler, 1999). This test presents 2×2 grids with geometric patterns in three of the four cells. Children see five answer choices and are asked to determine which pattern would be next in the sequence. Reliability is .90, and scores are correlated at .87 with overall nonverbal IQ.

Verbal ability. The Peabody Picture Vocabulary Test-IV (Dunn & Dunn, 2007) was used to control for children's general verbal ability. This test presents a sequence of words, and children are prompted to identify the corresponding picture from four choices. Reliability is .96 and scores are highly correlated (>.90) with overall verbal IQ (Dunn & Dunn, 2007).

Attentive behavior. Attentive behavior was measured using the teacher rating scale of the SWAN-I survey (Swanson et al., 2012). Nine items ($\alpha = .97$) gauge forms of inattentiveness commonly linked to attention-deficit hyperactivity disorder, according to the *Diagnostic and Statistical Manual of Mental Disorders* (4th ed.; American Psychiatric Association, 1994). Each student was rated on each item by his or her math teacher on a scale from 1 (*below average*) to 7 (*above average*).

Whole number line estimation (WNLE). To assess numerical magnitude knowledge, each student was presented with 22 number lines with endpoints of 0 and 100 and asked to mark the location of a given number (56, 606, 179, 122, 34, 78, 150, 938, 100, 163, 754, 5, 725, 18, 246, 722, 818, 738, 366, 2, 486, 147). Percent absolute error was calculated by averaging the absolute distance between each estimate and the correct value for all items and then scaling to calculate a percentage. Reliability for this task in third grade is high ($\alpha = .97$; Jordan et al., 2013).

Working memory. We assessed working memory using the Counting Recall subtest from the Working Memory Test Battery for Children (Pickering & Gathercole, 2001). Children were presented with sequences of dot arrays and were asked to recall (in order) how many dots were in each array. Following three correct responses in a span of six trials, the sequence was lengthened by one array. Test–retest reliability is relatively low at .61 (Pickering & Gathercole, 2001). As such, results associated with this measure should be interpreted cautiously.

Demographic variables. To control for demographic factors, we included dummy codes for gender (male = 1), ELL status (ELL = 1), and income status (free/reduced school lunch program participation; low income = 1), as well as age in months.

Procedures

Children completed tests of reading fluency and arithmetic fluency twice per year. Time points corresponded approximately to the winter of third grade, spring of third grade, fall of fourth grade, spring of fourth grade, fall of fifth grade, and winter of fifth grade. Children were assessed within a 1-month range for each time point. As noted previously, covariate measures were administered in third grade. All children were assessed individually in school.

Data Analysis

For each arithmetic operation, we built both bivariate latent growth models (LGMs) and autoregressive latent trajectory (ALT) models of the longitudinal relationship between arithmetic fluency and reading fluency. Examples of the structure of these models are shown in Figures 1 and 2, respectively. ALT models (Bollen & Curran, 2004; Bollen & Zimmer, 2010) offer several advantages over both LGMs and autoregressive cross-lagged (AR-CL) panel models. First, the LGM component uses the mean structure to reveal the overall shape of intraindividual (trait-like) trajectories of growth in reading fluency and arithmetic fluency as a function of time, and further uses the covariance structure to model individual deviations from the general trajectory of sample means, accounting for the effects of covariates related to general cognitive abilities or demographic factors. Second, the AR-CL component of the model can simultaneously capture more immediate, within-person lagged effects that are specific to the timing and content of educational curricula. The mathematics curriculum does not proceed in a uniform fashion, with topics receiving consistent attention over time. Rather, mathematics learning, in large part, reflects an in-

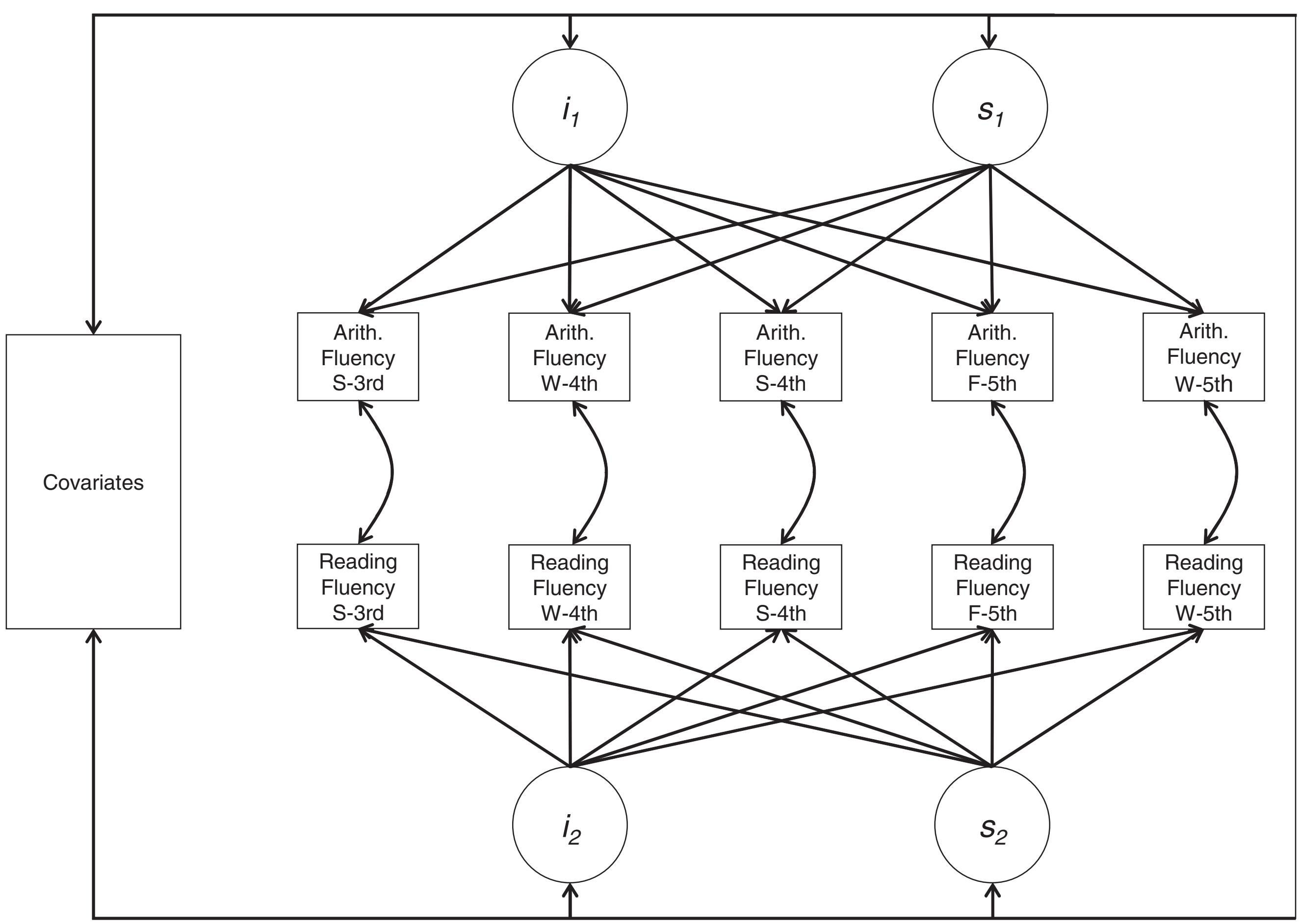

Figure 1. Example path diagram for a standard bivariate LGM with contemporaneous correlations and covariates. Arith. = arithmetic; i_n = intercept; s_n = slope; W = Winter; S = Spring; F = Fall. “3rd,” “4th,” and “5th” refer to grade levels.

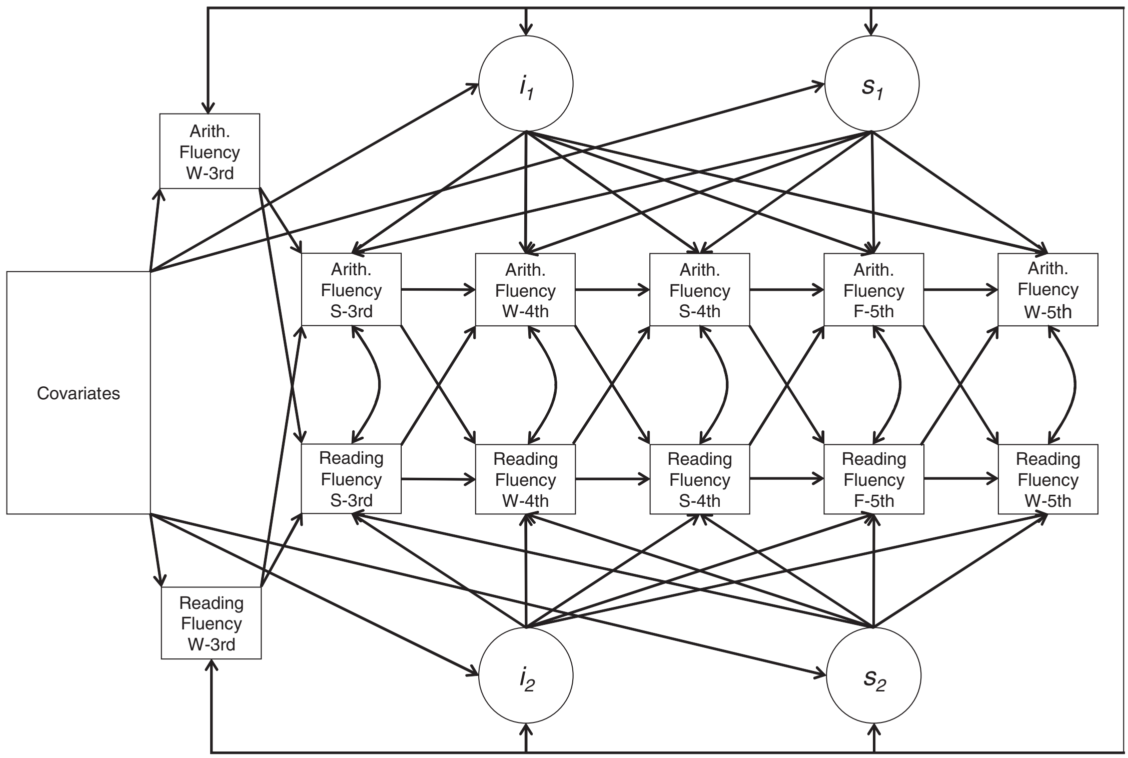

Figure 2. Example path diagram for a bivariate ALT model with covariates. Arith. = arithmetic; i_n = intercept; s_n = slope; W = Winter; S = Spring; F = Fall. "3rd," "4th," and "5th" refer to grade levels.

structional sequence during which children are taught one topic before moving on to the next, with different areas of mathematics receiving shifting degrees of attention over time. Another advantage of including auto-regressive cross-lagged effects is that instruction in mathematics (like all subjects) is interrupted by a very long summer break during which arithmetic practice naturally drops off. These breaks interrupt trends that might otherwise exhibit growth that is approximately linear or quadratic.

Although reading instruction may be more uniform over time, there are certainly similar changes in emphasis across schooling. Whereas early instruction may be focused on basic skills such as fluency, according to the CCSS, instruction later in elementary school focuses more on reading comprehension (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). Any effects of shifts in the focus of curricula would likely not be evident in a simple LGM but may be captured as a time-specific lagged effect in an AR-CL model.

By incorporating both bivariate LGM and AR-CL model components, ALT models can provide a more holistic picture of cognitive and academic development for multiple constructs in everyday school settings. Although it might be acceptable to assess growth in arithmetic fluency and its relation to reading fluency using LGMs that treat reading fluency as a time-varying covariate, such models could potentially obscure relationships between reading fluency growth factors and arithmetic fluency growth factors. Observing how these growth factors relate to one another may be interesting in its own right. In sum, the advantage of the bivariate ALT model for the present study is that it can capture overall trends in multiple dependent measures that may be subject to ongoing developmental processes as well as more punctuated effects associated with temporary instructional emphases and focused practice. If ALT models fit better than LGMs, this provides evidence that changes in reading and arithmetic fluency scores over time are influenced not only by latent growth factors but also by these shorter term autoregressive and/or cross-lagged effects.

To construct the most realistic models possible, we needed to account for the fact that test dates varied between individual students at each of the six test times, due primarily to limitations to the speed with which the team of researchers could collect data across multiple schools. Although testing children so frequently is a strength of our design, this also meant that the lengths of the intervals between time points varied not just from one time point to the next for the sample as a whole but also both within and across participants, potentially by as much as 2 months. For this reason, more traditional methods for analyzing panel data, such as repeated-measures ANOVA, would likely be inappropriate for the data we collected. To account for individually varying test times, we included random slopes in all LGMs and ALT models.

In addition to traditional LGMs and ALT models, we also built a "modified" version of each LGM in which the score for the first test time was treated as exogenous. These "modified" latent growth models were built primarily for the purpose of testing them against the fit of bivariate ALT models, as standard ALT models likewise treat the initial measurement of each dependent variable (i.e., "baseline" scores) as exogenous. Unlike the traditional latent growth model, this modified version is nested within the more complex ALT model, allowing for the use of likelihood ratio (LR) tests to compare model fit.

To determine the form of reading fluency growth, we first built a univariate LGM of reading fluency as well as a modified univariate LGM for subsequent inclusion in bivariate ALT models. After testing for linear growth in each model, we then tested models with a quadratic term. For the standard and modified models, respectively, we examined the Bayesian information criterion (BIC) to choose the models with the best-fitting reading fluency growth functions, respectively. Other fit indices for structural equation models were not available, because we included random effects to account for individually varying test times, and likelihood ratio tests are not appropriate for comparing linear versus quadratic growth models.[1] To understand the magnitude of differences in BIC, we follow the guidelines of Raftery (1995), who suggested that a BIC difference of 6 to 10 represents "strong" evidence in favor of the model with the lower BIC value (Bayes factor = 20–150), and a difference greater than 10 represents "very strong" evidence (Bayes factor >150). We then combined the best-fitting standard and modified reading fluency LGMs with analogous LGMs and ALT models for each arithmetic operation (addition, subtraction, and multiplication, respectively). All of these bivariate models included contemporaneous correlations between reading and math fluency at each test time, correlations among all growth factors, and, in the case of modified LGMs and ALT models, covariances among exogenous baseline scores and growth factors. Two versions of each bivariate LGM and ALT model were tested—first, a model with a linear growth term for arithmetic fluency, and then a model that additionally included a quadratic term.

To summarize, for each operation, the best-fitting bivariate longitudinal model linking reading fluency and arithmetic fluency (according to the BIC and/or LR tests) was selected from a set of six models: linear-only and quadratic versions, respectively, of simple bivariate latent growth models, modified bivariate LGMs with exogenous baseline scores, and bivariate ALT models. Simple AR-CL models were not examined because of issues of interpretation outlined by Hamaker, Kuiper, and Grasman (2015) and Voelkle (2008). Growth factors and/or exogenous baseline scores for the best-fitting model were then conditioned on the full set of background variables to create final models linking growth in reading fluency to growth in arithmetic fluency for each operation. All analyses were conducted using Mplus Version 7.1 (Muthén & Muthén, 2012) with full-information maximum likelihood (FIML) estimation. FIML estimation assigns values for missing data points based on the values of all other variables for the individual case in question. This maximizes power and reduces the chances of both Type I and Type II errors.

Results

Preliminary Results

Table 1 presents mean scores and standard deviations at each time point for our measure of reading fluency (TOWRE) as well as each operation subtest from the WIAT arithmetic fluency measure. A correlation matrix with all background predictors is shown in Table 2. An important detail evident in Table 1 is that the increase in multiplication scores across the first two time points (+5.89) is much greater than analogous increases for addition (+2.41) and subtraction (+2.90). This likely reflects the focus of instruction in third grade, when children are beginning to learn and practice multiplication but already have considerable experience with addition and subtraction computations (which therefore receive less attention). Although this preliminary descriptive evidence is informative, longitudinal models are required to more rigorously examine the codevelopment of reading fluency and arithmetic fluency across operations.

Longitudinal Models

In this section, we first present univariate models of reading fluency, followed by bivariate models of the relationship between reading fluency and arithmetic fluency for each operation (multiplication, addition, and subtraction). In each case, we selected the best-fitting model by examining BIC scores in conjunction with the guidelines described by Raftery (1995).

Reading fluency. Appendix A provides the full series of univariate LGM models of reading fluency tested. For the standard LGM, we found strong evidence in favor of the model with linear growth rather than quadratic growth (ΔBIC = 9.86). The mean intercept was 61.71 items correct ($p < .001$) and the variance was significant ($\sigma^2 = 88.68$, $p < .001$). The slope had a mean of 5.32 items per year ($p < .001$) and exhibited significant variance ($\sigma^2 = 2.55$, $p < .001$). For the modified version of the LGM model (in which the baseline score is exogenous), we found very strong evidence in favor of the model with linear growth (ΔBIC = 11.86). The values of the growth factors were nearly identical to those of the standard model; the mean intercept equaled 61.63 items correct ($p < .001$), with significant variance ($\sigma^2 = 87.14$, $p < .001$), and the mean slope was 5.36 items per year ($p < .001$) and also varied significantly ($\sigma^2 = 4.14$, $p = .004$). Thus, in the bivariate models of reading fluency and arithmetic fluency for each operation presented in the next three sections, growth in reading fluency is treated as linear.

To observe the effects of covariates on growth in reading fluency, we built a conditional version of the best-fitting unconditional model, which was the standard LGM with linear growth. Six of the nine covariates included in the model were significant predictors of the intercept term: nonverbal reasoning ($B = -0.326$, standard error [SE] = 0.156, $p = .037$), verbal ability ($B = .110$, $SE = 0.039$, $p = .005$), attentive behavior ($B = .252$, $SE = 0.040$, $p < .001$), WNLE error ($B = -0.208$, $SE = 0.092$, $p = .024$), low-income status ($B = -1.883$, $SE = .872$, $p = .031$), and age ($B = -.337$, $SE = .083$, $p < .001$). The only significant predictor of the reading fluency slope was low-income status ($B = .618$, $SE = .286$, $p = .031$). The slope and intercept terms did not covary significantly (estimate [est.] = −1.660, $SE = 1.190$, $p = .163$).

The results just presented show that verbal ability, attentive behavior, and WNLE error exhibited relatively predictable relationships to the intercept of growth in reading fluency, with the first two covariates having a positive effect and the third having

[1] Despite the fact that linear growth models are nested within quadratic growth models, likelihood ratio tests are not appropriate for comparing models with quadratic versus linear growth, because the variance of growth parameters is subject to a boundary condition. For more information, see Stoel, Garre, Dolan, and van den Wittenboer (2006).

Table 1
Means and Standard Deviations for Dependent Measures

Test time/ grade	TOWRE	WIAT multiplication	WIAT addition	WIAT subtraction
Winter-third	61.98 (10.39)	9.37 (6.61)	23.73 (6.57)	16.82 (8.00)
Spring-third	63.97 (10.26)	15.26 (6.59)	26.14 (6.64)	19.72 (8.36)
Winter-fourth	67.78 (9.94)	21.17 (6.84)	26.62 (7.03)	22.62 (8.46)
Spring-fourth	70.08 (10.17)	21.99 (7.21)	29.36 (6.34)	24.61 (8.65)
Fall-fifth	71.81 (10.44)	22.01 (7.02)	29.90 (6.98)	25.37 (8.73)
Winter-fifth	73.39 (10.07)	24.40 (7.24)	32.63 (6.86)	27.80 (9.04)

Note. TOWRE = Test of Word Reading Efficiency; WIAT = Wechsler Individual Achievement Test of arithmetic fluency.

the expected negative effect (lower error indicates better estimation accuracy). The negative effect of nonverbal reasoning on the intercept is surprising, but we believe this result may be spurious. The p value just reaches significance, and the effect may be influenced by the positive, albeit nonsignificant, effect of nonverbal reasoning on the slope term ($B = 0.041$). (All other things being equal, as the slope of growth increases, the intercept necessarily decreases.) The positive effect of low-income status on the slope of growth may also appear counterintuitive; however, this effect likely arose because students from low-income households exhibited significantly lower intercepts of growth, and over the period studied, these students were likely "catching up" with their peers from higher income households. Finally, the negative effect of age on the intercept term probably arises because children who are older than their peers are more likely to have started school late or have been held back as a result of poorer academic performance in earlier grades.

Multiplication fluency. Appendix B provides a correlation matrix that includes all dependent variables as well as covariates included in bivariate models of the relationship between reading fluency and multiplication fluency. (Note that pairwise correlations of covariates are not included in this matrix, which is why the lower right-hand corner of the matrix is not shown. These correlations were presented separately in Table 2.) Appendix C provides a full set of models tested with LR test results for nested pairs of models and BIC values for all models. We found very strong evidence that the best-fitting model overall was the bivariate ALT model with linear growth in multiplication fluency (ΔBIC = 33.7). Although the pattern of means in Table 1 could be taken to suggest a nonlinear trend in multiplication fluency over time, the change across the first two time points is better interpreted as a temporary difference in growth rate that is unique to the beginning of the study period. The subsequent trajectory from the spring of third grade to the winter of fifth grade reflects a more constant rate of growth. It is important to keep in mind that the first time point is exogenous to the growth function in ALT models; the latent growth component of the model is based strictly on values from the second time point (spring of third grade) onward, and a linear function therefore provides the best fit. The means and variances of the slope and intercept growth factors in the unconditional linear ALT model were all significant ($p < .001$). The mean slope indicated an increase of 4.66 items per year ($\sigma^2 = 3.44$), and the mean value of the intercept was 14.59 items correct ($\sigma^2 = 34.30$).

After establishing that the ALT with linear growth for multiplication fluency was the best-fitting unconditional model, we introduced the full set of covariates to construct a conditional model. Figure 3 gives the values of path coefficients in the AR-CL portion of this model; significant cross-domain covariances among growth factors and baseline reading fluency are highlighted. We found significant autoregressive path coefficients (ranging from .20 to .36) for multiplication fluency between third and fourth grade, the period during which multiplication fluency is typically acquired and honed. Across this stretch, multiplication fluency at each previous time point has a carryover effect on multiplication fluency at succeeding time points that is independent of the long-term trend captured by the latent growth function. Crucially, we also found a significant path coefficient of .11 from reading fluency in the winter of third grade (the first time point measured) to multi-

Table 2
Correlations Among Covariates

Covariate	1	2	3	4	5	6	7	8	9
Nonverbal reasoning	1								
Verbal ability	$.47^{**}$	1							
Attentive behavior	$.42^{**}$	$.38^{**}$	1						
Working memory	$.39^{**}$	$.28^{**}$	$.36^{**}$	1					
WNLE error	$-.38^{**}$	$-.39^{**}$	$-.40^{**}$	$-.36^{**}$	1				
Male	−.09	.05	$-.16^{**}$	.05	$-.20^{**}$	1			
Low income	$-.30^{**}$	$-.40^{**}$	$-.25^{**}$	$-.12^{*}$	$.17^{**}$	.00	1		
ELL	.04	$-.19^{**}$	.02	−.03	.05	.01	$.09^{*}$	1	
Age	$-.32^{**}$	−.06	$-.18^{**}$	$-.19^{**}$	$.15^{**}$	.05	$.16^{**}$	−.03	1

Note. $N = 449$ for all correlations. WNLE = whole number line estimation; ELL = English language learner.
$^{*}p < .05$. $^{**}p < .01$.

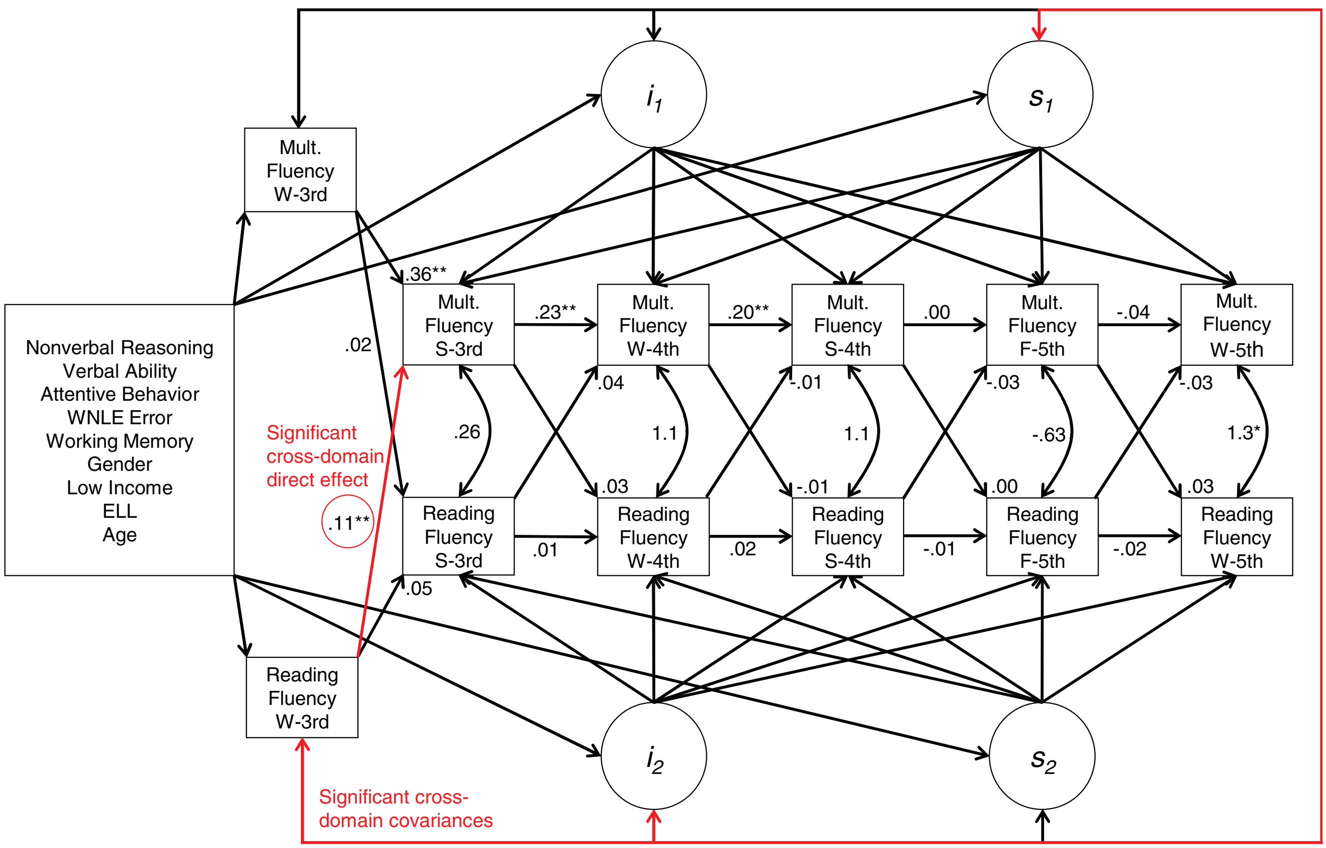

Figure 3. AR-CL weights and significant cross-domain covariance in ALT model of multiplication fluency. Mult. = multiplication; i_n = intercept; s_n = slope; W = Winter; S = Spring; F = Fall. “3rd,” “4th,” and “5th” refer to grade levels. * $p < .05$. ** $p < .01$. See the online article for the color version of this figure.

plication fluency in the spring of third grade; this path is also highlighted in Figure 3 and indicates the presence of a unidirectional cross-lagged effect of baseline reading fluency on the immediately subsequent measure of multiplication fluency, even after controlling for long-term growth trends.

In addition to significant results in the AR-CL portion of the model, we also found several significant covariances among baseline scores and growth factors. First, the baseline score, intercept, and slope of growth in multiplication fluency all significantly covaried with one another (all *p*s $< .001$). More importantly, the intercept of growth in reading fluency covaried significantly with the slope of multiplication fluency growth (est. = 6.524, *SE* = .665, $p < .001$), which indicates that early reading fluency scores predicted the long-term trajectory of multiplication fluency scores. Consistent with this finding, the baseline multiplication fluency score likewise covaried significantly with the slope of multiplication fluency growth (est. = 8.458, *SE* = .598, $p < .001$). Although the baseline score and growth intercept are conceptually similar, the corresponding parameters in the model differ, because the former is based on the slope of growth across the final five time points, whereas the latter is exogenous to the growth function. The fact that both predicted the slope of growth in multiplication fluency provides evidence of both an initial effect and an enduring effect of early reading fluency on growth in multiplication fluency.

Altogether, the significant results just reported represent strong evidence that reading fluency in early third grade predicts short-term growth in multiplication fluency between the beginning and end of third grade—during which time multiplication facts are initially taught—as well as longer-term growth between late third grade and fifth grade. The final bivariate model of multiplication fluency and reading fluency exhibits both kinds of effects captured by ALT models; we found both punctuated, time-dependent autoregressive and cross-lagged effects and effects of early reading fluency on the trajectory of trait-like long-term growth in multiplication fluency.

There were a number of significant covariate effects in the ALT model of multiplication fluency. As shown in Table 3, attentive behavior predicted both baseline multiplication fluency in the winter of third grade ($B = .16$) and the slope factor in the latent growth component of the model ($B = .10$). This suggests that multiplication fluency is associated with children’s effort and attention toward learning and practicing multiplication facts. Interestingly, WNLE percent absolute error (which we would expect to have a negative relationship to arithmetic fluency) significantly predicted both the intercept growth factor ($B = -.10$) and the slope factor ($B = -.07$). These results suggest that number line estimation accuracy—an indicator of the precision of magnitude representations—may be independently helpful for learning mul-

Table 3
Effects of Covariates on Multiplication Fluency Baseline, Growth Intercept, and Growth Slope

Predictor	Baseline (W-third)			Intercept (i_1)			Slope (s_1)		
	B	*SE*	*p*	*B*	*SE*	*p*	*B*	*SE*	*p*
Nonverbal reasoning	.09	.05	.115	.03	.08	.697	−.03	.06	.589
Verbal ability	.04	.04	.316	.00	.02	.835	.00	.02	.799
Attentive behavior	.16	.04	.001	.03	.02	.212	.10	.02	.001
Working memory	.04	.02	.094	.01	.01	.282	.01	.02	.594
WNLE error	−.16	.14	.241	−.10	.04	.007	−.07	.03	.034
Gender	1.52	.71	.031	.56	.41	.172	.25	.43	.571
Low income	−1.74	2.46	.479	−.77	.78	.320	−.13	.36	.725
ELL	2.77	1.59	.081	.93	.67	.167	−.04	.62	.953
Age	−.02	.04	.655	.01	.05	.755	−.07	.04	.039

Note. W-third = winter of third grade; *SE* = standard error; WNLE = whole number line estimation; ELL = English language learner.

tiplication facts. Finally, boys exhibited higher baseline multiplication fluency scores ($B = 1.52$), and there was a negative effect of age on the slope of multiplication fluency growth ($B = -.07$). The former effect suggests that boys may start out with greater proficiency before exhibiting a similar growth rate to females, but this result should be interpreted with caution, as an analogous effect was not observed for the intercept growth factor. Meanwhile, the latter effect of age suggests that older children improve more slowly than their peers over time, possibly because further improvement becomes difficult beyond a certain point. However, as noted for the univariate model of reading fluency, this effect could also arise from an increased likelihood that children who are older than their peers started school late or were held back because of prior academic struggles.

Addition fluency. Appendix D provides a correlation matrix linking dependent measures with background covariates, and Appendix E provides the full set of bivariate models tested. We found very strong evidence that the best-fitting model was a simple bivariate LGM with linear growth in addition fluency (ΔBIC = 12.3). The means and variances were significant for both the intercept and slope factors (*p*s $< .001$). For the intercept growth factor, the mean was 23.70 ($\sigma^2 = 30.50$). For the slope growth factor, the mean was 3.75 ($\sigma^2 = 2.793$). For purposes of parsimony, the conditional model was built by including covariates in this conventional bivariate LGM with linear growth rather than an ALT model. The fact that adding an AR-CL component did not improve the fit of the model shows that variability in addition and reading fluency scores over time is explained by trait-like growth characteristics but not time-specific autoregressive and cross-lagged effects.

In the final conditional LGM, the addition fluency intercept and slope of growth covaried significantly (est. = −2.631, *SE* = 0.212, $p < .001$); the fact that this relationship was negative may indicate that further improvement becomes difficult to achieve after a certain point. The growth intercepts of reading and addition fluency did covary significantly (est. = 11.510, *SE* = 2.420, $p < .001$), but neither the intercept nor the slope of reading fluency growth was significantly related to the slope of multiplication fluency growth (*p*s $> .74$). Thus, we found no evidence that reading fluency predicts growth in addition fluency.

Table 4 shows respective relations between covariates and the intercept and slope of addition fluency growth. Three covariates significantly predicted the value of the growth intercept: attentive behavior ($B = 0.10$), working memory ($B = -0.03$), and WNLE error ($B = -0.19$). The slope of growth in addition fluency was significantly related to attentive behavior ($B = 0.03$) and age ($B = -0.06$). Much like the negative effect of nonverbal reasoning on the intercept term in the univariate model of reading fluency, the negative effect of working memory found here also seems likely to be spurious; again, the *p* value is not particularly low, and the effect on the slope of growth is positive ($B = .01$), though not significant ($p = .219$). The effects of attentive behavior ($B = 0.10$) and WNLE error (−0.19), on the other hand, both exhibit the expected direction and are highly significant (*p*s $< .001$). As in the model of multiplication fluency, the negative effect of age on the slope of growth likely arose either because further improvement is difficult to achieve or because children with poorer math skills may be more likely to have started school late or repeated a grade.

Subtraction fluency. Appendix F shows pairwise correlations among subtraction fluency scores, reading fluency scores, and covariates, and Appendix G shows the full set of models tested. As with addition fluency, there was very strong evidence indicating that the best-fitting unconditional model overall was the conventional bivariate LGM with linear growth (ΔBIC = 21.5). Both growth factors for subtraction fluency growth exhibited significant means and variances (*p*s $< .001$). The mean intercept was 17.12

Table 4
Effects of Covariates on Addition Fluency Growth Intercept and Slope

Predictor	Intercept (i_1)			Slope (s_1)		
	B	*SE*	*p*	*B*	*SE*	*p*
Nonverbal reasoning	−.02	.11	.872	−.01	.05	.850
Verbal ability	−.02	.03	.484	.00	.01	.859
Attentive behavior	.10	.03	<.001	.03	.01	.031
Working memory	−.03	.02	.034	.01	.01	.219
WNLE error	−.19	.05	<.001	−.02	.03	.390
Gender	1.07	.58	.172	.04	.28	.901
Low income	−.25	.60	.680	−.19	.29	.507
ELL	−.70	.96	.462	.65	.51	.208
Age	.04	.06	.467	−.06	.03	.033

Note. *SE* = standard error; WNLE = whole number line estimation; ELL = English language learner.

items correct ($\sigma^2 = 55.53$), and the mean slope was an increase of 4.77 items correct per year ($\sigma^2 = 5.44$).

We added the full set of covariates to the standard bivariate LGM to build the final conditional model. As in the model of addition fluency, the slope and intercept of subtraction fluency growth covaried negatively (est. = −2.861, *SE* = 0.988, $p = .004$), and the intercept factors for reading and subtraction fluency covaried positively (est. = 11.785, *SE* = 2.441, $p < .001$). The slope of subtraction fluency growth did not covary significantly with either the reading fluency growth intercept (est. = 0.895, *SE* = 1.278, $p = .483$) or the slope (est. = −0.443, *SE* = 0.400, $p = .278$). Thus, as with addition fluency, there is no evidence that reading fluency predicts growth in subtraction fluency. However, a new and interesting relationship did emerge in the model of subtraction fluency—a significant covariance between the intercept of subtraction fluency growth and the slope of reading fluency growth (est. = 1.732, *SE* = 0.752, $p = .022$). This surprising result is at least suggestive of the possibility that some unmeasured common cognitive ability supports both reading fluency and subtraction fluency, whereas this does not appear to be the case for addition or multiplication fluency.

Effects of covariates in the final conditional model are shown in Table 5. The growth intercept was significantly predicted by verbal ability ($B = 0.08$), attentive behavior ($B = 0.18$), working memory ($B = 0.06$), WNLE error ($B = -0.19$), gender ($B = 1.71$), and low-income status ($B = -2.10$). The only significant predictor of the slope of growth was WNLE error ($B = -0.05$). All of these relationships exhibited the expected direction, and interestingly, the set of covariate effects on the subtraction fluency intercept better resemble those found in the model of reading fluency than those of the addition and multiplication fluency models. Considering that we also found a significant covariance between the intercept of subtraction fluency growth and the slope of reading fluency growth, the fact that verbal ability predicted growth intercepts for both subtraction fluency and reading fluency—but was unrelated to growth factors for multiplication and addition—points toward the possibility that subtraction fluency uniquely taps into abilities that also support reading.

Table 5
Effects of Covariates on Subtraction Fluency Growth Intercept and Slope

Predictor	Intercept (i_1) *B*	*SE*	*p*	Slope (s_1) *B*	*SE*	*p*
Nonverbal reasoning	.08	.12	.535	.03	.06	.671
Verbal ability	.08	.03	.005	−.02	.01	.112
Attentive behavior	.18	.03	<.001	.02	.01	.099
Working memory	.06	.02	.004	.00	.01	.985
WNLE error	−.19	.05	<.001	−.05	.03	.047
Gender	1.71	.62	.005	.29	.31	.349
Low income	−2.10	.68	.002	.18	.33	.584
ELL	.67	.95	.484	−.12	.59	.843
Age	−.02	.06	.762	−.06	.03	.065

Note. *SE* = standard error; WNLE = whole number line estimation; ELL = English language learner.

Discussion

The present study investigated the relationship between growth in arithmetic fluency and growth in reading fluency from third to fifth grade. We predicted that relations between reading fluency and arithmetic fluency would vary across operations because of differences in instructional timing and format associated with changes in concurrent reading ability as well as task demands. In contrast to addition or subtraction number combinations, products of single-digit factors are learned after basic reading skills have been acquired. Multiplication tables or lists of facts are typically memorized through repetitious practice and/or rehearsal, because alternative strategies for calculating products with even modestly sized factors are difficult. We expected that reading fluency would be more important for the development of multiplication fluency than for the development of addition or subtraction fluency.

Our expectation was confirmed by the observation of differences between the best-fitting model of multiplication fluency and best-fitting models of addition and subtraction fluency. Only for the model of multiplication fluency did the more complex bivariate ALT model fit better than a simpler bivariate LGM. The autoregressive cross-lagged effects unique to the model of multiplication fluency revealed a significant direct effect of early third-grade reading fluency on late third-grade multiplication fluency as well as effects of early reading fluency on the slope of growth in multiplication fluency in later grades. The former effect is present when efforts to memorize multiplication facts first begin and likely reflects short-term effects of instruction and practice. Moreover, initial reading fluency predicted the slope of growth in multiplication fluency, pointing to the additional presence of a sustained long-term effect of early reading fluency. Altogether, our results indicate that after controlling for verbal and nonverbal cognitive abilities as well as number line estimation accuracy, children with better reading fluency in the early grades have an advantage during the development of multiplication fluency that spans both initial multiplication instruction and growth in subsequent grades.

In contrast to Koponen et al.'s (2016) finding that general cognitive variables explain associations between reading fluency and arithmetic fluency, there does not appear to be a straightforward common-cause explanation for the relationships we observed between reading fluency and multiplication fluency. That is, because we controlled for a wide range of variables corresponding to verbal and nonverbal cognitive abilities as well whole number line estimation accuracy—an important predictor of later mathematics ability (e.g., Jordan et al., 2013)—it is unlikely that the unique statistical effects of early reading fluency on later growth in multiplication fluency are attributable to some unmeasured variable. A hypothetical unmeasured variable of this sort would have to produce an increase in early reading fluency—but not reading fluency growth—while also increasing both initial ability and growth in multiplication fluency. We argue instead that reading fluency directly affects multiplication fluency because of students' common practice of learning multiplication facts from written materials and repeatedly reading, reciting, and/or practicing responses. Because addition and subtraction facts are learned prior to the development of efficient reading skills, they are taught primarily through oral instruction, pictorial representations, and physical objects or manipulatives rather than symbolic arithmetic expressions. Also, instruction in addition and subtraction facts typically occurs before the ability to use explicit rehearsal strategies emerges, limiting children's ability to effortfully memorize arithmetic facts by rote (Henry & Millar, 1993).

Attentive behavior (as measured by teacher ratings in third grade) predicted growth in fluency for all operations, whereas whole number line estimation accuracy predicted growth in both multiplication and subtraction fluency. Effects of whole number line estimation ability suggest that precise representations of numerical magnitude support the strengthening of associations between number pairs and their differences and products. Addition fluency may be less dependent on the precision of magnitude representations, because counting strategies are reasonably efficient for solving simple addition problems. It may also be that addition facts have been learned well enough by third grade (the beginning of our study) that the precision of magnitude representations has lost predictive power.

A further commonality between the models of addition and subtraction fluency was that working memory predicted the intercept of fluency growth, a relation that was absent in the model of multiplication fluency. This result may reflect greater use of procedural (e.g., counting) strategies for solving addition and subtraction problems. Work by LeFevre, Bisanz, et al. (1996) and LeFevre, Sadesky, and Bisanz (1996) shows that for adults, use of procedural strategies yields greater response latencies for multiplication problems than for addition problems with similarly sized terms. Because the kinds of procedural strategies that tap strongly into working memory are much more difficult to execute for multiplication, working memory may be less predictive of multiplication fluency.

Verbal ability appeared to have a unique relationship to subtraction fluency that was not observed for addition and multiplication fluency. Verbal ability predicted the value of the growth intercept solely for the models of reading fluency and subtraction fluency, and the intercept of subtraction fluency growth predicted the slope of reading fluency growth. Together, these results suggest that there may be more overlap between subtraction fluency skills and general language ability than is typically recognized. Fluency with subtraction facts was generally poorer than fluency with addition facts; subtraction fluency performance was closer to (though still somewhat better than) multiplication fluency performance. One possible explanation is that subtraction facts are less transparent than addition facts, and procedural strategies (e.g., backward counting) are harder to implement accurately and quickly. Because children are taught subtraction before they can read effectively, memorization and fluency may be more dependent on possessing phonological verbal representations of individual subtraction facts. Future research should investigate this possibility.

In addition to effects of general cognitive variables like those highlighted by Koponen et al. (2016), our work points toward more direct effects of reading fluency on success at memorizing multiplication facts. Precise representations of whole number magnitudes may aid this process by helping students rule out subsets of erroneous products for a given pair of factors. For example, when trying to memorize the product 6×4, precise representation of the magnitudes of the numbers 6 and 4 might help limit interference from knowledge of other products and narrow the range of solutions that seem plausible (e.g., values larger than 10 but smaller than 40). Thus, whole number magnitude knowledge may help mitigate the kinds of working memory interference effects found by Ashkenazi et al. (2012).

However, if children have relatively weak numerical magnitude representations, multiplication fluency may become more difficult to attain than addition or subtraction fluency, as procedural strategies for solving multiplication problems are less efficient. So long as children know the basic counting sequence up to 20, there are relatively quick procedural ways to find correct responses to single-digit addition and subtraction problems, even if the sum or difference is not automatically known (e.g., finger counting, counting on from an addend; Baroody, 1985; Jordan, Kaplan, Ramineni, Locuniak, 2008). Fewer simple alternatives to direct fact retrieval are available for most multiplication problems. Thus, children may learn the full set of multiplication facts primarily by practicing responses to written expressions (e.g., flash cards, problem sets) or studying structured tables of facts, often learning facts for one factor at a time (e.g., 4s, 5s, 6s). If children struggle with reading fluency, they may be slower not only to read multiplication tables or lists of facts as they try to memorize them but also to read and respond to written problems while practicing, allowing for more decay of information in working memory and decreasing the number of rote repetitions a child can undertake in a given amount of time. As noted earlier, the number of repetitions of multiplication facts is a crucial factor in the acquisition of multiplication fluency, particularly for students who struggle in math (Burns et al., 2015). We believe that this confluence of circumstances leads to reading fluency being more important for multiplication fluency than for addition or subtraction fluency.

Effects of weak reading fluency on the development of multiplication fluency could also contribute to cases of comorbidity between reading and math difficulties that go unnoticed until children need to memorize a large number of multiplication facts in third grade. Relevant here is work by Chong and Siegel (2008) that distinguishes between "procedural deficits" and "fact fluency deficits." Both kinds of deficits undermine math performance, but Chong and Siegel showed that second graders with procedural deficits catch up to their typically achieving peers between second and fifth grade, whereas children with fact fluency deficits (i.e., an inability to immediately retrieve facts) made little to no progress relative to their peer group. In addition, children with fact fluency deficits also exhibited persistently poorer phonological processing of unfamiliar written words. Because a fact fluency deficit would naturally affect multiplication fluency more than addition and subtraction fluency, this result appears to converge with our finding that reading fluency is related to multiplication fluency growth but not addition and subtraction fluency growth. Research by Koponen et al. (2018) lends further support to this view; they observed stronger connections between math and reading difficulties in third and fourth grade than in first and second grade. Later identification of arithmetic fluency weaknesses in children who struggle with reading likely compounds the challenges of subsequent mathematics learning (e.g., acquisition of fraction skills that depend on multiplication fluency).

The results of the present study may have useful implications for instruction. Children with poor reading fluency may benefit from instruction in multiplication facts that does not depend solely on written representations (thereby limiting effects of reading difficulties). One alternative to using written materials may be to rely primarily on *oral* strategies for memorizing multiplication facts, at least initially. Although work by Paivio (1990) and others has shown that the use of both oral and visual representations supports dual-coding—which is typically viewed as superior for memorization—research on dyslexia (e.g., Palmer, 2000) suggests that for those with reading difficulties, holding information in working memory in both visual and verbal forms may be more of a challenge, depleting higher level cognitive resources that are necessary to achieve reading fluency. It is plausible that a similar phenomenon might occur when children with reading difficulties try to learn arithmetic facts.

If struggles with multiplication fluency are frequently accompanied by phonological processing weaknesses, it is possible that oral activities for teaching multiplication facts could have some disadvantages. However, by the time children are learning multiplication facts, they already know all of the words being uttered (number words, "times," etc.), so such activities may not tax phonological processing abilities very much. Simply repeating facts after a teacher—or an audio recording—may allow children to verbally encode multiplication facts without being slowed by the process of having to read them. Embedding multiplication facts in songs, rhymes, or raps may also help struggling readers to approach an otherwise tedious and difficult process (Gfeller, 1983). It is important, of course, that children ultimately recall and express multiplication facts using symbols, but this may be relatively trivial once the child already has a strong verbal memory trace for the product of a given pair of factors.

It is also interesting to consider cross-cultural differences in instruction in light of the present findings. Of particular interest is the primary method for learning multiplication facts in Japan—*ku-ku*, a rhythmic song or chant that students use to recite the multiplication table up to factors of nine without the use of a visual aid ("ku-ku" translates literally to "nine nines"). Beyond the potential benefits to multiplication fluency for Japanese children described by Dehaene (2011), it is noteworthy that *only* multiplication facts are taught this way and *not* addition or subtraction facts (Ito, Nagahara, & Hatta, 2004).

Several limitations of our study need to be addressed through further research. Foremost, our study is correlational, and we therefore cannot rule out the possibility that some unmeasured variable is confounding our results. However, we controlled for a wide variety of cognitive and demographic factors and are aware of no plausible alternative explanations for the differential effects of reading fluency we observed for multiplication fluency versus addition and subtraction fluency. Nonetheless, our findings should motivate the use of randomized controlled experiments to contrast different methods for learning multiplication facts (e.g., oral practice vs. written practice) among students with weak multiplication and reading fluency. A second limitation relates to the test of reading fluency used—we only measured children's speed at reading a list of words, and this may not be fully representative of reading speed for connected text that is linguistically similar to arithmetic expressions. However, speed and accuracy of reading single words is highly correlated with contextual oral reading rate (Eason, Sabatini, Goldberg, Bruce, & Cutting, 2013). A third limitation is that we were unable to account precisely for how multiplication facts were taught and reinforced in participating schools and classrooms. More research is needed to pinpoint how finer-grained differences in arithmetic instruction might affect student development. Finally, because multiplication is typically taught at least 2 years after addition and subtraction, children were older at the time of multiplication instruction, meaning that age effects could be at play. By the time children reach third grade, the reading curriculum has changed as well; children are likely no longer as deeply engaged in efforts to improve reading fluency, which could likewise contribute to differences across operations. These issues are largely unavoidable for any effort to measure the longitudinal bivariate relationship between reading fluency and arithmetic fluency.

To conclude, our study makes important strides toward understanding the dynamic processes through which children acquire arithmetic fluency. We have shown that it is inadequate to consider the development of arithmetic fluency in terms of a single fluency construct; the relationship to reading fluency is stronger for multiplication fact fluency than for addition and subtraction fact fluency, and addition and subtraction appear to differ from one another with respect to covariate effects. Researchers and educators should take care to discriminate among arithmetic operations when considering what factors contribute to arithmetic fluency and what teaching methods may be most useful for different populations of students. More work is needed, but our results nonetheless provide information that may be useful to educators right away. When reading fluency is lacking, students may benefit if they are presented with alternatives for learning multiplication facts that do not require as much reading. Such instruction may help stave off potential threats to long-term mathematics achievement.

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Appendix A

Comparison of Latent Growth Models of Reading Fluency

Model number	Type	Reading fluency baseline	Reading fluency growth function	BIC
1	Unconditional univariate LGM	Endogenous	Linear	16,246.841[a]
2	Unconditional univariate LGM	Endogenous	Quadratic	16,256.704
3	Modified unconditional univariate LGM	Exogenous	Linear	16,266.335[b]
4	Modified unconditional univariate LGM	Exogenous	Quadratic	16,278.199
5	Conditional univariate LGM	Endogenous	Linear	14,726.813

Note. BIC = Bayesian information criterion; LGM = Latent growth model.
[a] Best-fitting model with endogenous baseline score; model of reading fluency growth in bivariate LGMs of arithmetic fluency and reading fluency. [b] Best-fitting model with exogenous baseline score; model of reading fluency growth in bivariate autoregressive latent trajectory models of arithmetic fluency and reading fluency.

(*Appendices continue*)

Appendix B

Correlations (Sample Size) Between Multiplication Fluency Scores, Reading Fluency Scores, and Covariates

Variable	1	2	3	4	5	6	7	8	9	10	11	12
1. Mult. W-third	1 (426)											
2. Mult. S-third	.77** (401)	1 (422)										
3. Mult. W-fourth	.70** (366)	.80** (362)	1 (386)									
4. Mult. S-fourth	.66** (358)	.75** (357)	.87** (378)	1 (379)								
5. Mult. F-fifth	.64** (327)	.71** (326)	.83** (344)	.85** (344)	1 (347)							
6. Mult. W-fifth	.65** (327)	.74** (326)	.83** (343)	.87** (342)	.87** (344)	1 (347)						
7. Read. W-third	.40** (421)	.47** (419)	.51** (382)	.5** (376)	.49** (344)	.50** (344)	1 (444)					
8. Read. S-third	.36** (426)	.44** (422)	.47** (386)	.46** (379)	.42** (347)	.45** (347)	.85** (444)	1 (449)				
9. Read. W-fourth	.38** (364)	.46** (360)	.49** (384)	.47** (377)	.45** (344)	.45** (343)	.85** (380)	.82** (384)	1 (384)			
10. Read. S-fourth	.38** (358)	.46** (357)	.47** (378)	.49** (379)	.46** (344)	.47** (342)	.83** (376)	.84** (379)	.88** (377)	1 (379)		
11. Read. F-fifth	.39** (327)	.48** (326)	.47** (344)	.47** (344)	.46** (347)	.46** (344)	.84** (344)	.79** (347)	.87** (344)	.87** (344)	1 (347)	
12. Read. W-fifth	.40** (327)	.47** (326)	.48** (343)	.48** (342)	.48** (344)	.50** (347)	.84** (344)	.79** (347)	.85** (343)	.86** (342)	.89** (344)	1 (347)
13. Nonverbal reasoning	.35** (426)	.3** (422)	.29** (386)	.27** (379)	.28** (347)	.27** (347)	.27** (444)	.19** (449)	.26** (384)	.20** (379)	.28** (347)	.28** (347)
14. Verbal ability	.35** (426)	.28** (422)	.28** (386)	.22** (379)	.26** (347)	.24** (347)	.35** (444)	.30** (449)	.32** (384)	.30** (379)	.36** (347)	.35** (347)
15. Attentive behavior	.47** (426)	.43** (422)	.45** (386)	.45** (379)	.52** (347)	.47** (347)	.45** (444)	.40** (449)	.42** (384)	.43** (379)	.45** (347)	.45** (347)
16. Working memory	.33** (426)	.29** (422)	.29** (386)	.28** (379)	.30** (347)	.32** (347)	.23** (444)	.26** (449)	.22** (384)	.20** (379)	.24** (347)	.25** (347)
17. WNLE error	−.42** (426)	−.40** (422)	−.43** (386)	−.41** (379)	−.44** (347)	−.43** (347)	−.36** (444)	−.32** (449)	−.26** (384)	−.31** (379)	−.33** (347)	−.34** (347)
18. Male	.11* (426)	.11* (422)	.08 (386)	.07 (379)	.05 (347)	.07 (347)	.04 (444)	−.01 (449)	−.03 (384)	.00 (379)	.01 (347)	.02 (347)
19. Low income	−.29** (426)	−.25** (422)	−.23** (386)	−.18** (379)	−.18** (347)	−.18** (347)	−.25** (444)	−.25** (449)	−.21** (384)	−.18** (379)	−.23** (347)	−.19** (347)
20. ELL	.11* (426)	.07 (422)	.04 (386)	.04 (379)	.02 (347)	.02 (347)	−.02 (444)	−.04 (449)	−.05 (384)	−.07 (379)	−.09 (347)	−.06 (347)
21. Age	−.16** (426)	−.15** (422)	−.22** (386)	−.23** (379)	−.22** (347)	−.23** (347)	−.26** (444)	−.24** (449)	−.27** (384)	−.23** (379)	−.27** (347)	−.25** (347)

Note. Mult. = Multiplication; Read. = Reading; WNLE = whole number line estimation; ELL = English language learner; W = Winter; S = Spring; F = Fall. "Third," "fourth," and "fifth" refer to grade levels.
* $p < .05$. ** $p < .01$.

(*Appendices continue*)

Appendix C

Comparison of Bivariate Models of Reading Fluency and Multiplication Fluency

Model	Type	Multiplication fluency baseline	Multiplication fluency growth function	LR test	BIC
1	Unconditional bivariate LGM	Endogenous	Linear		31,228.847
2	Unconditional bivariate LGM	Endogenous	Quadratic	N/A	30,892.970
3	Modified unconditional bivariate LGM	Exogenous	Linear		30,818.173
4	Unconditional bivariate ALT	Exogenous	Linear	$\chi^2(16) = 64.79, p < .001$	30,756.935[a]
5	Modified unconditional bivariate LGM	Exogenous	Quadratic		30,799.284
6	Unconditional bivariate ALT	Exogenous	Quadratic	$\chi^2(16) = 125.82, p < .001$	30,790.680
Final	Conditional bivariate ALT	Exogenous	Linear		27,927.730

Note. LR = likelihood ratio; BIC = Bayesian information criterion; LGM = Latent growth model; N/A = not applicable; ALT = autoregressive latent trajectory model.
[a] Best-fitting unconditional model; basis for final conditional model.

(*Appendices continue*)

Appendix D

Correlations (Sample Size) Between Addition Fluency Scores, Reading Fluency Scores, and Covariates

Variable	1	2	3	4	5	6	7	8	9	10	11	12
1. Add. W-third	1 (426)											
2. Add. S-third	.71** (401)	1 (422)										
3. Add. W-fourth	.52** (366)	.54** (362)	1 (386)									
4. Add. S-fourth	.68** (358)	.69** (357)	.67** (378)	1 (379)								
5. Add. F-fifth	.61** (327)	.62** (326)	.65** (344)	.73** (344)	1 (347)							
6. Add. W-fifth	.65** (327)	.64** (326)	.66** (343)	.76** (342)	.76** (344)	1 (347)						
7. Read. W-third	.36** (421)	.35** (419)	.36** (382)	.40** (376)	.42** (344)	.42** (344)	1 (444)					
8. Read. S-third	.32** (426)	.38** (422)	.35** (386)	.41** (379)	.39** (347)	.37** (347)	.85** (444)	1 (449)				
9. Read. W-fourth	.32** (364)	.36** (360)	.35** (384)	.40** (377)	.41** (344)	.42** (343)	.85** (380)	.82** (384)	1 (384)			
10. Read. S-fourth	.36** (358)	.39** (357)	.35** (378)	.44** (379)	.43** (344)	.42** (342)	.83** (376)	.84** (379)	.88** (377)	1 (379)		
11. Read. F-fifth	.34** (327)	.38** (326)	.32** (344)	.42** (344)	.42** (347)	.40** (344)	.84** (344)	.79** (347)	.87** (344)	.87** (344)	1 (347)	
12. Read. W-fifth	.37** (327)	.41** (326)	.35** (343)	.43** (342)	.42** (344)	.45** (347)	.84** (344)	.79** (347)	.85** (343)	.86** (342)	.89** (344)	1 (347)
13. Nonverbal reasoning	.19** (426)	.15** (422)	.23** (386)	.20** (379)	.21** (347)	.29** (347)	.27** (444)	.19** (449)	.26** (384)	.20** (379)	.28** (347)	.28** (347)
14. Verbal ability	.18** (426)	.15** (422)	.16** (386)	.19** (379)	.24** (347)	.23** (347)	.35** (444)	.30** (449)	.32** (384)	.30** (379)	.36** (347)	.35** (347)
15. Attentive behavior	.31** (426)	.27** (422)	.32** (386)	.37** (379)	.41** (347)	.44** (347)	.45** (444)	.40** (449)	.42** (384)	.43** (379)	.45** (347)	.45** (347)
16. Working memory	.26** (426)	.21** (422)	.24** (386)	.28** (379)	.30** (347)	.34** (347)	.23** (444)	.26** (449)	.22** (384)	.20** (379)	.24** (347)	.25** (347)
17. WNLE error	−.34** (426)	−.29** (422)	−.36** (386)	−.38** (379)	−.39** (347)	−.44** (347)	−.36** (444)	−.32** (449)	−.26** (384)	−.31** (379)	−.33** (347)	−.34** (347)
18. Male	.13** (426)	.07 (422)	.06 (386)	.08 (379)	.10 (347)	.07 (347)	.04 (444)	−.01 (449)	−.03 (384)	.00 (379)	.01 (347)	.02 (347)
19. Low income	−.13** (426)	−.07 (422)	−.14** (386)	−.14** (379)	−.16** (347)	−.11* (347)	−.25** (444)	−.25** (449)	−.21** (384)	−.18** (379)	−.23** (347)	−.19** (347)
20. ELL	−.02 (426)	−.04 (422)	.03 (386)	−.01 (379)	.00 (347)	.02 (347)	−.02 (444)	−.04 (449)	−.05 (384)	−.07 (379)	−.09 (347)	−.06 (347)
21. Age	−.07 (426)	−.04 (422)	−.17** (386)	−.12* (379)	−.18** (347)	−.18** (347)	−.26** (444)	−.24** (449)	−.27** (384)	−.23** (379)	−.27** (347)	−.25** (347)

Note. Add. = addition; Read. = reading; WNLE = whole number line estimation; ELL = English language learner; W = Winter; S = Spring; F = Fall. "Third," "fourth," and "fifth" refer to grade levels.
* $p < .05$. ** $p < .01$.

(*Appendices continue*)

Appendix E

Comparison of Bivariate Models of Reading Fluency and Addition Fluency

Model	Type	Addition fluency baseline	Addition fluency growth function	LR test	BIC
1	Unconditional bivariate LGM	Endogenous	Linear		31646.151[a]
2	Unconditional bivariate LGM	Endogenous	Quadratic	N/A	31662.096
3	Modified unconditional bivariate LGM	Exogenous	Linear		31658.485
4	Unconditional bivariate ALT	Exogenous	Linear	$\chi^2(16) = 66.32$, $p < .001$	31680.223
5	Modified unconditional bivariate LGM	Exogenous	Quadratic		31705.306
6	Unconditional bivariate ALT	Exogenous	Quadratic	$\chi^2(16) = 139.30$, $p < .001$	31705.157
Final	Conditional bivariate LGM	Endogenous	Linear		28688.077

Note. LR = Likelihood ratio; BIC = Bayesian information criterion; LGM = latent growth model; N/A = not applicable; ALT = autoregressive latent trajectory model.
[a] Best-fitting unconditional model; basis for final conditional model.

(*Appendices continue*)

Appendix F

Correlations (Sample Size) Between Subtraction Fluency Scores, Reading Fluency Scores, and Covariates

Variable	1	2	3	4	5	6	7	8	9	10	11	12
1. Sub. W-third	1											
	(426)											
2. Sub. S-third	.85**	1										
	(401)	(422)										
3. Sub. W-fourth	.75**	.78**	1									
	(366)	(362)	(386)									
4. Sub. S-fourth	.75**	.79**	.84**	1								
	(358)	(357)	(378)	(379)								
5. Sub. F-fifth	.75**	.79**	.85**	.87**	1							
	(327)	(326)	(344)	(344)	(347)							
6. Sub. W-fifth	.72**	.76**	.82**	.86**	.9**	1						
	(327)	(326)	(343)	(342)	(344)	(347)						
7. Read. W-third	.44**	.47**	.44**	.48**	.51**	.49**	1					
	(421)	(419)	(382)	(376)	(344)	(344)	(444)					
8. Read. S-third	.41**	.48**	.41**	.45**	.46**	.44**	.85**	1				
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)				
9. Read. W-fourth	.43**	.49**	.42**	.45**	.46**	.44**	.85**	.82**	1			
	(364)	(360)	(384)	(377)	(344)	(343)	(380)	(384)	(384)			
10. Read. S-fourth	.44**	.50**	.42**	.48**	.48**	.47**	.83**	.84**	.88**	1		
	(358)	(357)	(378)	(379)	(344)	(342)	(376)	(379)	(377)	(379)		
11. Read. F-fifth	.44**	.51**	.43**	.48**	.48**	.46**	.84**	.79**	.87**	.87**	1	
	(327)	(326)	(344)	(344)	(347)	(344)	(344)	(347)	(344)	(344)	(347)	
12. Read. W-fifth	.47**	.53**	.45**	.50**	.50**	.49**	.84**	.79**	.85**	.86**	.89**	1
	(327)	(326)	(343)	(342)	(344)	(347)	(344)	(347)	(343)	(342)	(344)	(347)
13. Nonverbal reasoning	.35**	.34**	.35**	.35**	.36**	.37**	.27**	.19**	.26**	.20**	.28**	.28**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
14. Verbal ability	.42**	.37**	.36**	.38**	.40**	.35**	.35**	.30**	.32**	.30**	.36**	.35**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
15. Attentive behavior	.49**	.44**	.46**	.49**	.50**	.52**	.45**	.40**	.42**	.43**	.45**	.45**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
16. Working memory	.37**	.34**	.34**	.35**	.34**	.37**	.23**	.26**	.22**	.20**	.24**	.25**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
17. WNLE error	−.44**	−.41**	−.47**	−.46**	−.47**	−.49**	−.36**	−.32**	−.26**	−.31**	−.33**	−.34**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
18. Male	.11*	.13**	.12*	.10	.11*	.09	.04	−.01	−.03	.00	.01	.02
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
19. Low income	−.34**	−.27**	−.23**	−.26**	−.31**	−.22**	−.25**	−.25**	−.21**	−.18**	−.23**	−.19**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
20. ELL	−.03	.01	.02	−.02	−.02	−.03	−.02	−.04	−.05	−.07	−.09	−.06
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)
21. Age	−.16**	−.13**	−.19**	−.19**	−.24**	−.24**	−.26**	−.24**	−.27**	−.23**	−.27**	−.25**
	(426)	(422)	(386)	(379)	(347)	(347)	(444)	(449)	(384)	(379)	(347)	(347)

Note. Sub. = subtraction; Read. = reading; WNLE = whole number line estimation; ELL = English language learner; W = Winter; S = Spring; F = Fall. "Third," "fourth," and "fifth" refer to grade levels.
* $p < .05$. ** $p < .01$.

(*Appendices continue*)

Appendix G

Comparison of Bivariate Models of Reading Fluency and Subtraction Fluency

Model	Type	Subtraction fluency baseline	Subtraction fluency growth function	LR test	BIC
1	Unconditional bivariate LGM	Endogenous	Linear		31,540.451[a]
2	Unconditional bivariate LGM	Endogenous	Quadratic	N/A	31,561.912
3	Modified unconditional bivariate LGM	Exogenous	Linear		31,573.694
4	Unconditional bivariate ALT	Exogenous	Linear	$\chi^2(16) = 59.69$, $p < .001$	31,590.015
5	Modified unconditional bivariate LGM	Exogenous	Quadratic		31,605.268
6	Unconditional bivariate ALT	Exogenous	Quadratic	$\chi^2(16) = 68.51$, $p < .001$	31,632.538
Final	Conditional bivariate LGM	Endogenous	Linear		28,489.993

Note. LR = likelihood ratio; BIC = Bayesian information criterion; LGM = latent growth model; N/A = not applicable; ALT = autoregressive latent trajectory model.
[a] Best-fitting unconditional model; basis for final conditional model.

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